CC/M/EXAM. 2020

MATHEMATICS

PAPER—I

Time : 3 hours] [Full Marks : 250

Note : Question Nos. **1** and **5** are compulsory and out of the remaining, any **three** are to be attempted choosing at least ONE question from each Section. The number of marks carried by a question/part is indicated against it.

SECTION—A

1. Answer *any* **five** of the following questions : $10 \times 5 = 50$

(a) Consider the linear transformation $T : \mathbb{R}^2(\mathbb{R}) \to \mathbb{R}^2(\mathbb{R})$ defined by $T(x, y) = (x, 0)$. Determine the matrix of T with respect to the standard basis $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

(b) Find the point at which the line joining the points $(2, -3, 1)$ and $(3, -4, -5)$ intersects the plane $2x + y + z = 7$.

(c) Show that $\int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta = \frac{\pi}{8} \log 2$.

(d) Find the equation of the sphere which passes through the origin and makes equal intercepts of unit length of the axes.

(e) Evaluate $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}}$.

(f) If $ax + by + cz = p$, then show that the minimum value of $x^2 + y^2 + z^2$ is $\frac{p^2}{a^2 + b^2 + c^2}$.

(g) Prove that determinant of a Hermitian matrix is real.

2. Answer the following questions :

(a) Suppose A is an $n \times n$ square matrix and $f(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of A. Show that $f(A) = 0$. 20

(b) Let W be the subspace of $\mathbb{R}^4(\mathbb{R})$ spanned by the vectors $(1,-2,5,-3)$, $(0,1,1,4)$ and $(1,0,1,0)$. Find a basis for W and extend it to a basis for $\mathbb{R}^4(\mathbb{R})$. 15

(c) Show that any system of linear equations $Ax = b$ has either no solution, exactly one solution or infinitely many solutions. 15

3. Answer the following questions :

(a) If V is a function of x and y, prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$, where $x = r\cos\theta$ and $y = r\sin\theta$. 20

(b) Find the area of the portion of the circle $x^2 + y^2 = 1$, which lies inside the parabola $y^2 = 1 - x$. 15

(c) Suppose a and b are real numbers such that $b > a > 0$. Show that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ and deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$. 15

4. Answer the following questions :

(a) Show that the condition that the plane $ax + by + cz = 0$ may intersect the cone $yz + zx + xy = 0$ in perpendicular lines, is given by $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. 20

(b) Find the equation of the curve in which the plane $z = h$ intersects the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and find the area enclosed by the curve. 15

(c) Show that the four points $(0,-1,0)$, $(2,1,-1)$, $(1,1,1)$ and $(3,3,0)$ are coplanar and obtain the equation of the plane containing the four points. 15

SECTION—B

5. Answer *any* **five** of the following questions : 10×5=50

(a) Find the integrating factors of the following first order differential equation :

$$\frac{dy}{dx}+\frac{y}{1+x}=2-\frac{y}{x}$$

(b) If the Inverse Laplace transform is defined as $L^{-1}\{f(s)\}=F(x)$, then evaluate the Inverse Laplace transform $L^{-1}\left\{\frac{s+1}{s^2+2s}\right\}$.

(c) Given that $p=\frac{dy}{dx}$, find the general solution of the differential equation

$$xp^2-2yp+5x=0$$

(d) If $\phi(x,y,z)=x^2yz+xy^2z-xyz^2$, find the gradient of ϕ at (1, 0, 1).

(e) A particle executing simple harmonic motion has velocities 0·8 m/sec and 0·6 m/sec when it is at the distance of 0·3 m and 0·4 m respectively from the mean position. Calculate the amplitude of the simple harmonic motion of the particle.

(f) Given that $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{A}=5\hat{i}-2\hat{j}+4\hat{k}$ and $r=|\vec{r}|$, evaluate $\vec{A}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$.

(g) How high can a particle rest inside a hollow sphere of radius a, if the coefficient of friction is given by $\frac{1}{\sqrt{3}}$?

6. Answer the following questions :

(a) Apply the method of variation of parameter to solve

$$\frac{d^2y}{dx^2}+9y=\sec 3x$$ 20

(b) Find the complementary function and the particular integral of the second order ordinary differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=3+2\sin 2x$$ 15

(c) Find the orthogonal trajectories of the cardioid defined by $r=a(1-\cos\theta)$, with a as a parameter. 15

7. Answer the following questions :

(a) An object starts from rest to move in a straight line with a uniform acceleration of a_1 to attain a speed v after an interval of time. It then suffers a uniform deceleration of a_2 along the same straight line to finally come to rest. If the object covers a total distance of S during the entire motion, show that the total time taken by the object to cover the distance is

$$\frac{S}{v} + \frac{v}{2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)$$ 20

(b) A stone of mass 0·5 kg is released from a height h under the action of earth's gravity. Find the amount of work done by the gravity during the 10th second of the stone's free fall. (Take acceleration due to gravity, $g = 9 \cdot 8$ m/sec^2) 15

(c) Two equal uniform rods AB and AC, each of length $2b$ are freely joined at A. The system rests on a smooth vertical circle of radius a. If the angle between the rods is 2θ, show that $b \sin^3 \theta = a \cos \theta$. 15

8. Answer the following questions :

(a) Using Stokes theorem, evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and C is the boundary of the triangle with vertices at $(0, 0, 0)$, $(1, 0, 0)$ and $(1, 1, 0)$. 20

(b) Find the vector equation of a plane which passes through a given point with position vector $\vec{a}$ and is parallel to the vectors $\vec{b}$ and $\vec{c}$. 15

(c) Given that $\vec{A}(x, y, z) = x^2 y\hat{i} - 2xy^3 z\hat{j} + yz\hat{k}$, find $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ at $(2, 0, -1)$. 15

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