ARUNACHAL PRADESH PUBLIC SERVICE COMMISSION

MATHEMATICS: PAPER - I

Time: 3 hours | Full marks 100

(Group – A is compulsory. Attempt any FOUR questions from Group – B).

GROUP – A
(This Group is COMPULSORY)

1: Attempt any ten (10) questions: 2 x 10=20

(i) The function $f(x) = |x - 1|$ is
- (a) continuous at $x = 1$
- (b) not derivable at $x = 1$
- (c) both (a) and (b) are true
- (d) none of these

(ii) The value of the integral $\int \frac{e^t}{1+e^t} dt =$
- (a) $1 + e^t$
- (b) $\log(1 + e^t)$
- (c) $[\log(1 + e^t)]^{-1}$
- (d) $[1 + e^t]^{-1}$

(iii) The set **N** of natural numbers is
- (a) bounded
- (b) bounded below
- (c) bounded above
- (d) not bounded

(iv) Every convergent sequence is
- (a) bounded
- (b) Cauchy
- (c) both (a) and (b) true
- (d) none of these

(v) A subgroup H of a group G is normal if and only if
- (a) $xH \subset Hx^{-1}, \forall x \in G$
- (b) $xH \subseteq Hx, \forall x \in G$
- (c) $xHx^{-1} \supset H, \forall x \in G$
- (d) $xHx^{-1} = H, \forall x \in G$

(vi) The set $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \dots\}$ is a
- (a) closed subset of R,
- (b) open subset of R,
- (c) both closed and open subset of R,
- (d) neither closed nor open subset of R.

(vii) Let $G = \{-1, 1\}$ be a group under multiplication. Define $f : Z \to G$ by

$f(n) = \begin{cases} 1, \text{ if } n \text{ is even;} \\ -1, \text{ if } n \text{ is odd} \end{cases}$, where (Z, +) is the group of integers. Then

(a) f is an isomorphism
(b) f is an endomorphism
(c) f is an automorphism
(d) f is an epimorphism

(viii) The set $S = \{a + ib, c + id\}$ is a basis of the space $C(R)$ if and only if

(a) $ad - bc \neq 0$
(b) $ab - cd \neq 0$
(c) $ac - bd \neq 0$
(d) $ad - bc = 0$

(ix) Consider the mapping $T : R^2 \to R^2$ such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$. Then

(a) $T(x, y) = (x + y, x - 2y)$
(b) $T(x, y) = (x + y, x + 2y)$
(c) $T(x, y) = (x - y, x - 2y)$
(d) $T(x, y) = (x - y, x + 2y)$

(x) The matrix $A = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$ is invertible if

(a) $\alpha = \pm 1$
(b) $\alpha = \pm i$
(c) $\alpha \neq \pm 1$
(d) none of these

(xi) The eigen values of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are

(a) 1, 6
(b) 2, 3
(c) 3, 4
(d) 4, 6

(xii) The fuzzy set operations satisfy

(a) Law of contradiction,
(b) Law of excluded middle,
(c) Both law of contradiction and law of excluded middle
(d) None of the above property.

(xiii) The following linear programming problem:

Maximize $z = x + y$ subject to the constraints:

$x + y \leq 1, -3x + y \geq 3, x \geq 0, y \geq 0$ has

(a) a unique optimal solution,
(b) infinite number of optimal solution
(c) no feasible solution
(d) unbounded solution.

(xiv) The bilinear transformation which maps the points $z = \infty, i, 0$ into the points $w = 0, i, \infty$, respectively is

(a) $w = \frac{1}{z}$

(b) $w = -\frac{1}{z}$

(c) $w = \frac{2}{z}$

(d) $w = -\frac{2}{z}$

GROUP – B

(Answer any Four.)

2: Attempt any two (2) questions:

(a) (i) Find the value of $\lim_{x \to 0}\left(\frac{1-\cos x}{x^2}\right)$. (5)

(ii) Show that $\int_0^\infty \frac{\sin x}{x^p}dx$ converges absolutely it $p > 1$. (5)

(b) Let $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ (10)

Show that $f_{xy} \neq f_{yx}$ at (0, 0). Also find f_{xx}, f_{yy} at (1, 1).

(c) Prove that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx = \beta(m,n)$. (10)

3: Attempt any two (2) questions:

(a) (i) Let R be a ring and let $a \in R$. Show that the subset $X = \{r \in R : ar = 0\}$ is a left ideal of R. (2)

(ii) Prove that every finite group G is isomorphic to a permutation group. (8)

(b) (i) Let Q^+ be the set of positive rational numbers. Define an operation $*$ on Q^+ as $a * b = \frac{ab}{3}$ for $a, b \in Q^+$. Verify that $(Q^+, *)$ is an abelian group. (5)

(ii) Prove that every field is an integral domain. (5)

(c) (i) If every element of a group G is its own inverse, then show that G is abelian. (5)

(ii) Prove that the intersection of two ideals is an ideal. (5)

4: Attempt any two (2) questions:

(a) (i) Show that the three vectors (1, 1, –1), (2, –3, 5) and (–2, 1, 4) of R^3 are linearly independent. (5)

(ii) Define rank of a matrix. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$. (5)

OR

(iii) Show that the union of two subspaces is a subspace if and only if one is contained in the other. (5)

(b) If U, W are two subspaces of a finite dimensional vector space $V(F)$, then prove that $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$. (10)

(c) Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for the matrix. Also find A^{-1}. (3+4+3=10)

5: Attempt any two (2) questions:

(a) (i) Show that the function $e^x(\cos y + i \sin y)$ is analytic. (5)

(ii) If a function $f(z)$ is continuous on a contour C of length L and if M is the upper bound of f(z) on C, prove that $\left|\int_C f(z)dz\right| \leq ML$. (5)

(b) State and prove Cauchy's Residue theorem. (10)

(c) State Cauchy's integral formula. Evaluate: $\int_C \frac{\sin^8 z}{\left(z-\frac{\pi}{6}\right)^4} dz$ if C is a circle $|z| = 1$. (2+8=10)

6: Attempt any two (2) questions: 10 x 2=20

(a) Solve the following LPP:

Maximize $z = 6x + 4y$ subject to the constraints:

$2x + 3y \leq 30,\ 3x + 2y \leq 24,\ x + y \geq 3,\ x \geq 0,\ y \geq 0.$

(b) Let (X, d) and (Y, ρ) be metric spaces. Prove that a function $f: X \to Y$ is continuous at $x_0 \in X$ iff $x_n \to x_0$ in X implies $f(x_n) \to f(x_0)$ in Y.

(c) Prove that a continuous image of a connected set is connected. Also, show that the property Hausdorff space is a hereditary property.

(d) Suppose a random variable X takes on the values $-3, -1, 2,$ and 5 with respective probabilities

$$\frac{2k-3}{10}, \frac{k-2}{10}, \frac{k-1}{10}, \frac{k+1}{10}$$

(i) Determine the distribution of X;

(ii) (ii) Find the expected value $E(X)$ of X.

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