ARUNACHAL PRADESH PUBLIC SERVICE COMMISSION

MATHEMATICS: PAPER-II

Time: 3 (Three) Hours — Full marks - 100

(Group – A is compulsory. Attempt any FOUR questions from Group – B).

GROUP – A

(This group is COMPULSORY)

1. Attempt any ten (10) questions: **2x10=20**

   (a) What is the maximum number of edges in a simple graph with n vertices?

   (b) A and B are two events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$, and $P(A^c) = \frac{2}{3}$, find $P(A \cap B^c)$.

   (c) For any integer $n > 1$, find all possible values of $g.c.d.(n-1, n+1)$.

   (d) What do mean by stream line of a fluid particle?

   (e) What do you mean by Internet?

   (f) Write down the Cauchy-Riemann equations for an analytic function $f(z) = u + iv$.

   (g) Write down the value of the integral $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2} dz$.

   (h) What are the conditions that a set S is a basis of a vector space V(R).

   (i) Let Q be the set of all rational numbers. What is the derived set of Q ?

   (j) What is the norm of the bounded sequence space ?

   (k) Write down the value of p for which the space ℓ_p is not a Hilbert space.

   (l) Give a suitable condition such that the linear operator T on a Hilbert space H is unitary.

GROUP – B

(Answer any Four.)

2. Attempt any two (2) questions:

   (a) (i) Prove that every convergent sequence in a normed space is Cauchy. (5)

   (ii) In a Hilbert space H, prove that $x \perp y$ if and only if (5)

   $$\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2 \text{ for all } x, y \in H.$$

(b) State and prove closed graph theorem. **(10)**

(c) Let H be a Hilbert space. If x, y be any two vectors in H, then prove that

(i) $|\langle x,y\rangle| \le \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$ **(6)**

(ii) $\|x+y\|^2 - \|x-y\|^2 = 4\,\mathrm{Re}\langle x,y\rangle$. **(4)**

3. Attempt any two (2) questions:

(a)

(i) If a graph G with p vertices and q edges is self complementary, then show that $p \equiv 0 \text{ or } 1 (mod\ 4)$. **(3)**

(ii) Define a triangulation graph. Show that the numbers of edges in a triangulation graph of order n is $3n - 6$. **(5)**

(iii) What is the minimum number of vertices necessary for a graph with six edges to be planar? **(2)**

(b) Show that any graph with n vertices is a tree if and only if it has $n - 1$ edges. **(10)**

(c) If X is a Poisson variable with mean m, show that $Z = \frac{X-m}{\sqrt{m}}$ is a variable with mean zero and variance unity. Find the moment generating function for the variable Z and show that it approaches $exp(t^2/2)$ as $m \longrightarrow \infty$. Also interpret the result. **(10)**

4. Attempt any four (4) questions:

(a) Use Hamilton's equations to find the equations of motion of a projectile. **(5)**

(b) State and prove d'Alembert's principle. **(5)**

(c) Show that the motion of a body about its centre of inertia is the same as it would be if the centres of inertia were fixed and the same forces acted on the body. **(5)**

(d) A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined plane at an angle α, to the horizon, and a man of mass M', starting from the upper end, walks down the plank so that it does not move; show that he gets to the other end in time, **(5)**

$$\sqrt{\left[\frac{2M'a}{(M+M')g\ sin\alpha}\right]}$$

Where a is the length of the plank.

(e) Find the MI of a right circular cylinder about (i) its axis, (ii) a straight line through the centre of gravity perpendicular to its axis. **(5)**

(f) Derive Hamilton's canonical equations. **(5)**

5. Attempt any four (4) questions:

(a) Derive the differential equation with initial conditions $y(0) = 1$ and $y'(0) = -2$ from the integral equation **(5)**

$$y(x) = 1 - x - 4sinx + \int_0^x [3 - 2(x-t)y(t)]dt$$

(b) Solve

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}$$

Subject to $U(0,t) = 0, U(5,t) = 0$, where $U(x,0) = 10sin4\pi x$, is bounded for $x > 0, t > 0$. **(5)**

(c) Find $F(x)$ if its Fourier sine transform is e^{-as}/s. Hence, deduce $F_s^{-1}(1/s)$. **(5)**

(d) Using Fourier integral, show that **(5)**

$$e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{cos\lambda x}{\lambda^2 + a^2} dx \ , \quad a > 0 \ , \ x \geq 0.$$

(e) If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then prove that $\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F(t/k)$. **(5)**

6. Attempt any two (2) questions:

(a) (i) Solve:

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{x(x^2-y^2)}$$ **(5)**

(ii) Find f(z) such that

$$(y^2 + z^2 - x^2)dx - 2xydy + 2xf(z)dz = 0$$

is integrable and hence solve it. **(5)**

(b) Find the solution of Bessel's equation $xy'' + y' + xy = 0$ in a series for $x = 0$. **(10)**

(c) (i) Reduce the differential equation $y'' + Py' + Qy = R$, where P, Q and R are functions of x to the normal form **(6)**

$$\frac{d^2V}{dx^2} + IV = S.$$

(ii) Discuss the existence and uniqueness solution for the IVP **(4)**

$$y' = \frac{2y}{x}, \qquad y(x_0) = y_0$$

7. Attempt any four (4) questions:

(a) If the velocity of an incompressible fluid at the point (x, y, z) is given by

$$\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5}$$

Prove that the liquid motion is possible and that the velocity potential is **(5)**

$$\frac{cos\theta}{r^2}.$$

(b) Two sources, each of strength m are placed at the points (-a, 0) and (a, 0) and a sink of strength 2m at the origin. Show that stream lines are the curves **(5)**

$$(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$$

Where λ is a variable parameter.

(c) Illustrate a forward error analysis. What is backward error analysis? **(5)**

(d) Define an ill-conditioned matrix. Explain it with an example. What do you mean by condition number? **(5)**

(e) The following table for y_i and y_i' at points x_i is given:

x	:	-1	0	1
$y(x)$	:	1	1	3
$y'(x)$	:	-5	1	7

Approximate $y(0.5)$ using Hermite interpolating polynomial. (5)

(f) (i) Write short notes on Compiler and Interpreter. (2)

(ii) Given the following declarations:

$$int\ x = 11,\ \ y = 20;$$
$$int\ *p1 = \&x,\ \ *p2 = \&y;$$

What is the value of each of the following expressions? (3)

$$(*p1++)\ ;\ --(*p2)\ ;\ *p1 + (*p2)$$

8. Attempt any two (2) questions:

(a) (i) The mean and variance of a binomial variable X are 4 and $\frac{4}{3}$, respectively. Find $P(X \geq 1)$. (2)

(ii) Joint distribution of X and Y is given by (3)

$$f(x,y) = xy\, e^{-(x^2+y^2)},\ \text{for } x \geq 0 \text{ and } y \geq 0.$$

Find the conditional density function of X given $Y = y$.

(iii) For n events $A_1, A_2, \cdots, A_n$, prove that (5)

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

(b) (i) Use Jacobi's symbol to test whether solution of the congruence $x^2 \equiv 16(mod\ 65)$ exists or not? (2)

(ii) Find the remainder when 41^{75} is divided by 3. (3)

(iii) A boy has 100 rupees which he has to spend in buying two fruits: Apple and Mango. The cost of an apple is Rs. 10 and a mango is Rs. 15. If the boy has to buy both the fruits, in how many different ways he can spend the 100 rupees. (5)

(c) Prove that the linear congruence $ax \equiv b(mod\ m)$ has a solution if and only if d divides b, where $d = g.c.d.(a,m)$. Further show that $ax \equiv b(mod\ m)$ has d incongruent modulo m solutions which are expressed in the form $x_0 + r\frac{m}{d}$, for $0 \leq r \leq d-1$, where x_0 is an arbitrary solution. (10)

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