CC/M/EXAM.
2020

STATISTICS

PAPER—I

Time : 3 hours] [Full Marks : 250

Note : Question Nos. **1** and **5** are compulsory and out of the remaining, any **three** are to be attempted choosing at least ONE question from each Section. The number of marks carried by a question/part is indicated against it.

SECTION—A

1. Answer *any* **five** of the following questions : 10×5=50

(a) Let X_1 and X_2 be independent and normally distributed with mean 0 and variance 1. Prove that $Y = \frac{X_1}{X_2}$ is Cauchy distributed.

(b) The variables X and Y are distributed normally with zero means and variances σ_1^2 and σ_2^2 respectively and correlation coefficient ρ. Let $U = \frac{X}{\sigma_1} + \frac{Y}{\sigma_2}$ and $V = \frac{X}{\sigma_1} - \frac{Y}{\sigma_2}$. Show that X and Y are independent normal variates with variances $2(1+\rho)$ and $2(1-\rho)$ respectively.

(c) Show for a BIBD having v treatments in b blocks of k plots each, where each treatment is replicated r times and each pair of treatment occurs together in λ blocks

$$\lambda(v-1) = r(k-1)$$

(d) Show that $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ in random sampling from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & ; 0 < x < \infty; 0 < \theta < \infty \\ 0, \text{ otherwise} \end{cases}$$

is an MVB estimate of θ and has variance $\frac{\theta^2}{n}$.

(e) Suppose a researcher has coded Doctor's response into two categories A and B over a small group of 10 mentally retarded patients. The patients are allocated randomly to one of the two categories and after a few months, a test was conducted for each patient. The score obtained were as follows :

Category A :	110	125	89	90	48
Category B :	60	52	33	10	28

Is there any significant difference among the scores in the two categories? (Table value of U for equal sample sizes for $n = 5$ at 5% level of significance is 2.)

(f) Construct a 2^5 design in blocks of 8 plots each in a replicate, confounding the interactions ABC and BDE, mentioning the steps of obtaining the blocks. So, identify any other effect which gets confounded in this layout.

(g) Consider the covariance matrix of a bivariate population given by

$$\operatorname{cov}\begin{pmatrix} X_1^{(1)} \\ X_2^{(1)} \\ X_1^{(2)} \\ X_2^{(2)} \end{pmatrix} = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 100 & 0 & 0 & 0 \\ 0 & 1 & 0\cdot 95 & 0 \\ \hline 0 & 0\cdot 95 & 1 & 0 \\ 0 & 0 & 0 & 100 \end{array}\right]$$

Verify that the first pair of canonical variates are $U_1 = X_2^{(1)}, V_1 = X_1^{(2)}$ with canonical correlation $\rho_1^* = 0\cdot 95$.

2. Answer the following questions :

(a) Discuss Wilcoxon Signed Rank test and derive the mean and variance of the Wilcoxon Signed Rank test statistic T^+ under $H_0 : \mu_m = \mu_m{}^0$ (μ_m = population median). Also, show that under H_0, T^+ is symmetric about its mean. 20

(b) Explain ratio and regression estimator of the population mean and find their variances. Also, compare the efficiency of the two estimators with that of the estimator obtained under SRSWOR scheme. 15

(c) Show that the mean deviation about mean of the Chi-square distribution with n d.f. is given by

$$\frac{\Gamma\left(\frac{n}{2}\right) e^{\frac{-n}{2}} n^{\frac{n}{2}}}{2^{\frac{n-4}{2}}}$$ 15

3. Answer the following questions :

(a) State the classical linear regression model and mention its assumptions along with their detection and consequences of their violation. Also, carry out the least square estimation of the parameters of the model. 2(

(b) A sample survey is to be undertaken to ascertain the mean annual income of farms in a certain area. The farms are stratified according to their principal products. A Census conducted several years earlier revealed the following information :

Type of farm	**No. of farms (N_i)**	**Mean annual income ($\bar{Y}_{N_i}$)**	**Standard deviation (σ_i)**
Sheep	161	10946	2236
Wheat	195	6402	2614
Dairy	274	2228	606
Others	382	1458	230

For a sample of 12 farms, determine the sample size in each stratum under proportional allocation and Neyman's optimum allocation. Also, obtain the variance of the estimate of the population mean under both proportional and optimum allocation and compare the precision of both these estimates with that of the estimate obtained under SRSWOR. 15

(c) Let $x_1, x_2, \ldots, x_n$ be i.i.d. random variables from uniform $(0, \theta)$ distribution, where $\theta > 0$. Find the maximum likelihood estimate of θ and verify if the estimate is unbiased and consistent for θ. 15

4. Answer the following questions :

(a) Discuss in detail the split plot design and its statistical analysis. 20

(b) Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from $N(\theta, 100)$. For testing $H_0 : \theta = 75$ against $H_1 : \theta > 75$, the following test procedure is proposed :

$$\text{Reject } H_0 \text{ if } \bar{x} \geq c$$

$$\text{Accept } H_0 \text{ if } \bar{x} < c$$

Determine n and c so that the power function $P(\theta)$ of the test satisfies

$$P(75) = 0 \cdot 159 \text{ and } P(77) = 0 \cdot 841$$ 15

(c) Derive the characteristic function of the $N_p(\mu, \Sigma)$ distribution with $|\Sigma| > 0$. 15

SECTION—B

5. Answer *any* **five** of the following questions : 10×5=50

(a) Using Chebyshev's inequality, prove that for $n > 36$, the probability that in n throws of a fair die, the number of sixes that lie between $\frac{n}{6} - \sqrt{n}$ and $\frac{n}{6} + \sqrt{n}$ is at least $\frac{31}{36}$.

(b) Let X_1, X_2 and X_3 be three uncorrelated r.v.'s having common variance σ^2. If $E(X_1) = \theta_1 + \theta_2$, $E(X_2) = 2\theta_1 + \theta_2$, $E(X_3) = \theta_1 + 2\theta_2$, show that the least square estimates $\hat{\theta}_1, \hat{\theta}_2$ of θ_1, θ_2 satisfy the equations

$$6\hat{\theta}_1 + 5\hat{\theta}_2 = x_1 + 2x_2 + x_3$$

$$5\hat{\theta}_1 + 6\hat{\theta}_2 = x_1 + x_2 + 2x_3$$

Hence, show that $\hat{\theta}_1 = \dfrac{x_1 + 7x_2 - 4x_3}{11}$, $\hat{\theta}_2 = \dfrac{x_1 - 4x_2 + 7x_3}{11}$ are unbiased estimates.

(c) Find the efficiency of LSD relative to RBD.

(d) Estimate α and β by the method of moments for the distribution

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}; 0 \le x < \infty$$

(e) Show that if $X_3 = aX_1 + bX_2$, the three partial correlations $r_{12\cdot3}$, $r_{13\cdot2}$ and $r_{23\cdot1}$ are numerically equal to unity. Further, $r_{13\cdot2}$ is having the sign of a, $r_{23\cdot1}$, the sign of b and $r_{12\cdot3}$, the opposite sign of $\dfrac{a}{b}$.

(f) Box I contains 3 red and 5 white balls while Box II contains 4 red and 2 white balls. A ball is chosen at random from the first box and placed in the second box without observing its colour. Then a ball is drawn from the second box. Find the probability that it is white.

(g) Let $X_1, X_2 \ldots, X_n$ be n independent $N(\mu, \sigma^2)$ variables where μ is unknown but σ^2 is known. If the prior distribution of μ is $N(\theta, \sigma_0^2)$, find the Bayes estimate of μ assuming squared error loss function.

6. Answer the following questions :

(a) State and prove the Lehmann–Scheffe's theorem. Hence, find the UMVUE of $g(\lambda) = e^{-\lambda}(1+\lambda)$, where $x_1, x_2, \ldots, x_n$ is a random sample from Poisson (λ). 2(

(b) Throw some light on the Horvitz–Thompson estimator of the population mean. Show that it is unbiased for the population mean and also find its variance. 1!

(c) Consider a one-way classified data on a independent groups or samples, where the number of observations under each group is b. Show that the expected value of the between group variation V_B and the within group variation V_W (assuming random effect model) are given by

$$E(V_B) = (a-1)\sigma^2 + b\sum_{j=1}^{a} \alpha_j^2$$

$$E(V_W) = a(b-1)\sigma^2$$

where α_j is the jth treatment effect, $j = 1, 2, \ldots, a$ and σ^2 is such that each error term $\varepsilon_{ij} (i = 1, 2, \ldots, a; j = 1, 2, \ldots, b)$ is normally distributed with mean zero and variance σ^2. 15

7. Answer the following questions :

(a) State and prove the Kolmogorov's strong law of large numbers and the Lindeberg-Levy central limit theorem. 20

(b) Find the unbiased estimator of the population mean of a variable, say X per second-stage in a two-stage sampling with each first-stage unit having equal number of second-stage units. Also, find the variance of the estimator. 15

(c) Construct the likelihood ratio test for comparing the means of k normal populations with common variance. 15

8. Answer the following questions :

(a) The number of labourers x (in thousands) and the quantity of raw materials y (in lakhs of bales) are given below for 15 jute mills :

Serial No.	x	y
1	368	31
2	384	33
3	361	37
4	347	39
5	403	43
6	529	61
7	703	68
8	396	42
9	473	41
10	509	49
11	512	31
12	503	29
13	472	38
14	429	41
15	387	40

Draw a sample of 5 units with PPS taking x as the size and hence estimate the total amount of raw materials consumed by the 15 mills using Lahiri's method. Also, give an estimate of its standard error. 20

(b) Construct the SPRT for testing $H_0 : \theta = \theta_0$ against $H_0 : \theta = \theta_1 \;(> \theta_0)$ in sampling from a normal density

$$f(x; \theta) = \frac{1}{2\pi} e^{-\frac{1}{2}(x-\theta)^2}$$

Also, obtain its O.C. function and A.S.N. function. 15

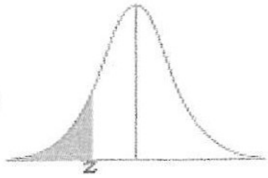
(c) In a sequence of Bernoulli trails, let X be the length of the run of either successes or failures starting with the first trial. Find $E(X)$ and $V(X)$. 15

Table of Random Digits

00000	10097	32533	76520	13586	34673	54876	80959	09117	39292	74945
00001	37542	04805	64894	74296	24805	24037	20636	10402	00822	91665
00002	08422	68953	19645	09303	23209	02560	15953	34764	35080	33606
00003	99019	02529	09376	70715	38311	31165	88676	74397	04436	27659
00004	12807	99970	80157	36147	64032	36653	98951	16877	12171	76833
00005	66065	74717	34072	76850	36697	36170	65813	39885	11199	29170
00006	31060	10805	45571	82406	35303	42614	86799	07439	23403	09732
00007	85269	77602	02051	65692	68665	74818	73053	85247	18623	88579
00008	63573	32135	05325	47048	90553	57548	28468	28709	83491	25624
00009	73796	45753	03529	64778	35808	34282	60935	20344	35273	88435
00010	98520	17767	14905	68607	22109	40558	60970	93433	50500	73998
00011	11805	05431	39808	27732	50725	68248	29405	24201	52775	67851
00012	83452	99634	06288	98083	13746	70078	18475	40610	68711	77817
00013	88685	40200	86507	58401	36766	67951	90364	76493	29609	11062
00014	99594	67348	87517	64969	91826	08928	93785	61368	23478	34113
00015	65481	17674	17468	50950	58047	76974	73039	57186	40218	16544
00016	80124	35635	17727	08015	45318	22374	21115	78253	14385	53763
00017	74350	99817	77402	77214	43236	00210	45521	64237	96286	02655
00018	69916	26803	66252	29148	36936	87203	76621	13990	94400	56418
00019	09893	20505	14225	68514	46427	56788	96297	78822	54382	14598
00020	91499	14523	68479	27686	46162	83554	94750	89923	37089	20048
00021	80336	94598	26940	36858	70297	34135	53140	33340	42050	82341
00022	44104	81949	85157	47954	32979	26575	57600	40881	22222	06413
00023	12550	73742	11100	02040	12860	74697	96644	89439	28707	25815

Standard Normal Cumulative Probability Table

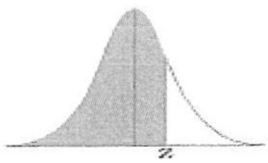
Cumulative probabilities for NEGATIVE z-values are shown in the following table :



z	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
–3·4	0·0003	0·0003	0·0003	0·0003	0·0003	0·0003	0·0003	0·0003	0·0003	0·0002
–3·3	0·0005	0·0005	0·0005	0·0004	0·0004	0·0004	0·0004	0·0004	0·0004	0·0003
–3·2	0·0007	0·0007	0·0006	0·0006	0·0006	0·0006	0·0006	0·0005	0·0005	0·0005
–3·1	0·0010	0·0009	0·0009	0·0009	0·0008	0·0008	0·0008	0·0008	0·0007	0·0007
–3·0	0·0013	0·0013	0·0013	0·0012	0·0012	0·0011	0·0011	0·0011	0·0010	0·0010
–2·9	0·0019	0·0018	0·0018	0·0017	0·0016	0·0016	0·0015	0·0015	0·0014	0·0014
–2·8	0·0026	0·0025	0·0024	0·0023	0·0023	0·0022	0·0021	0·0021	0·0020	0·0019
–2·7	0·0035	0·0034	0·0033	0·0032	0·0031	0·0030	0·0029	0·0028	0·0027	0·0026
–2·6	0·0047	0·0045	0·0044	0·0043	0·0041	0·0040	0·0039	0·0038	0·0037	0·0036
–2·5	0·0062	0·0060	0·0059	0·0057	0·0055	0·0054	0·0052	0·0051	0·0049	0·0048
–2·4	0·0082	0·0080	0·0078	0·0075	0·0073	0·0071	0·0069	0·0068	0·0066	0·0064
–2·3	0·0107	0·0104	0·0102	0·0099	0·0096	0·0094	0·0091	0·0089	0·0087	0·0084
–2·2	0·0139	0·0136	0·0132	0·0129	0·0125	0·0122	0·0119	0·0116	0·0113	0·0110
–2·1	0·0179	0·0174	0·0170	0·0166	0·0162	0·0158	0·0154	0·0150	0·0146	0·0143
–2·0	0·0228	0·0222	0·0217	0·0212	0·0207	0·0202	0·0197	0·0192	0·0188	0·0183
–1·9	0·0287	0·0281	0·0274	0·0268	0·0262	0·0256	0·0250	0·0244	0·0239	0·0233
–1·8	0·0359	0·0351	0·0344	0·0336	0·0329	0·0322	0·0314	0·0307	0·0301	0·0294
–1·7	0·0446	0·0436	0·0427	0·0418	0·0409	0·0401	0·0392	0·0384	0·0375	0·0367
–1·6	0·0548	0·0537	0·0526	0·0516	0·0505	0·0495	0·0485	0·0475	0·0465	0·0455
–1·5	0·0668	0·0655	0·0643	0·0630	0·0618	0·0606	0·0594	0·0582	0·0571	0·0559
–1·4	0·0808	0·0793	0·0778	0·0764	0·0749	0·0735	0·0721	0·0708	0·0694	0·0681
–1·3	0·0968	0·0951	0·0934	0·0918	0·0901	0·0885	0·0869	0·0853	0·0838	0·0823
–1·2	0·1151	0·1131	0·1112	0·1093	0·1075	0·1056	0·1038	0·1020	0·1003	0·0985
–1·1	0·1357	0·1335	0·1314	0·1292	0·1271	0·1251	0·1230	0·1210	0·1190	0·1170
–1·0	0·1587	0·1562	0·1539	0·1515	0·1492	0·1469	0·1446	0·1423	0·1401	0·1379
–0·9	0·1841	0·1814	0·1788	0·1762	0·1936	0·1711	0·1685	0·1660	0·1635	0·1611
–0·8	0·2119	0·2090	0·2061	0·2033	0·2005	0·1977	0·1949	0·1922	0·1894	0·1867
–0·7	0·2420	0·2389	0·2358	0·2327	0·2296	0·2266	0·2236	0·2206	0·2177	0·2148
–0·6	0·2743	0·2709	0·2676	0·2643	0·2611	0·2578	0·2546	0·2514	0·2483	0·2451
–0·5	0·3085	0·3050	0·3015	0·2981	0·2946	0·2912	0·2877	0·2843	0·2810	0·2776
–0·4	0·3446	0·3409	0·3372	0·3336	0·3300	0·3264	0·3228	0·3192	0·3156	0·3121
–0·3	0·3821	0·3783	0·3745	0·3707	0·3669	0·3632	0·3594	0·3557	0·3520	0·3483
–0·2	0·4207	0·4168	0·4129	0·4090	0·4052	0·4013	0·3974	0·3936	0·3897	0·3859
–0·1	0·4602	0·4562	0·4522	0·4483	0·4443	0·4404	0·4364	0·4325	0·4286	0·4247
–0·0	0·5000	0·4960	0·4920	0·4880	0·4840	0·4801	0·4761	0·4721	0·4681	0·4641

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table :



z	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
0·0	0·5000	0·5040	0·5080	0·5120	0·5160	0·5199	0·5239	0·5279	0·5319	0·5359
0·1	0·5398	0·5438	0·5478	0·5517	0·5557	0·5596	0·5636	0·5675	0·5714	0·5753
0·2	0·5793	0·5832	0·5871	0·5910	0·5948	0·5987	0·6026	0·6064	0·6103	0·6141
0·3	0·6179	0·6217	0·6255	0·6293	0·6331	0·6368	0·6406	0·6443	0·6480	0·6517
0·4	0·6554	0·6591	0·6628	0·6664	0·6700	0·6736	0·6772	0·6808	0·6844	0·6879
0·5	0·6915	0·6950	0·6985	0·7019	0·7054	0·7088	0·7123	0·7157	0·7190	0·7224
0·6	0·7257	0·7291	0·7324	0·7357	0·7389	0·7422	0·7454	0·7486	0·7517	0·7549
0·7	0·7580	0·7611	0·7642	0·7673	0·7704	0·7734	0·7764	0·7794	0·7823	0·7852
0·8	0·7881	0·7910	0·7939	0·7967	0·7995	0·8023	0·8051	0·8078	0·8106	0·8133
0·9	0·8159	0·8186	0·8212	0·8238	0·8264	0·8289	0·8315	0·8340	0·8365	0·8389
1·0	0·8413	0·8438	0·8461	0·8485	0·8508	0·8531	0·8554	0·8577	0·8599	0·8621
1·1	0·8643	0·8665	0·8686	0·8708	0·8729	0·8749	0·8770	0·8790	0·8810	0·8830
1·2	0·8849	0·8869	0·8888	0·8907	0·8925	0·8944	0·8962	0·8980	0·8997	0·9015
1·3	0·9032	0·9049	0·9066	0·9082	0·9099	0·9115	0·9131	0·9147	0·9162	0·9177
1·4	0·9192	0·9207	0·9222	0·9236	0·9251	0·9265	0·9279	0·9292	0·9306	0·9319
1·5	0·9332	0·9345	0·9357	0·9370	0·9382	0·9394	0·9406	0·9418	0·9429	0·9441
1·6	0·9452	0·9463	0·9474	0·9484	0·9495	0·9505	0·9515	0·9525	0·9535	0·9545
1·7	0·9554	0·9564	0·9573	0·9582	0·9591	0·9599	0·9608	0·9616	0·9625	0·9633
1·8	0·9641	0·9649	0·9656	0·9664	0·9671	0·9678	0·9686	0·9693	0·9699	0·9706
1·9	0·9713	0·9719	0·9726	0·9732	0·9738	0·9744	0·9750	0·9756	0·9761	0·9767
2·0	0·9772	0·9778	0·9783	0·9788	0·9793	0·9798	0·9803	0·9808	0·9812	0·9817
2·1	0·9821	0·9826	0·9830	0·9834	0·9838	0·9842	0·9846	0·9850	0·9854	0·9857
2·2	0·9861	0·9864	0·9868	0·9871	0·9875	0·9878	0·9881	0·9884	0·9887	0·9890
2·3	0·9893	0·9896	0·9898	0·9901	0·9904	0·9906	0·9909	0·9911	0·9913	0·9916
2·4	0·9918	0·9920	0·9922	0·9925	0·9927	0·9929	0·9931	0·9932	0·9934	0·9936
2·5	0·9938	0·9940	0·9941	0·9943	0·9945	0·9946	0·9948	0·9949	0·9951	0·9952
2·6	0·9953	0·9955	0·9956	0·9957	0·9959	0·9960	0·9961	0·9962	0·9963	0·9964
2·7	0·9965	0·9966	0·9967	0·9968	0·9969	0·9970	0·9971	0·9972	0·9973	0·9974
2·8	0·9974	0·9975	0·9976	0·9977	0·9977	0·9978	0·9979	0·9979	0·9980	0·9981
2·9	0·9981	0·9982	0·9982	0·9983	0·9984	0·9984	0·9985	0·9985	0·9986	0·9986
3·0	0·9987	0·9987	0·9987	0·9988	0·9988	0·9989	0·9989	0·9989	0·9990	0·9990
3·1	0·9990	0·9991	0·9991	0·9991	0·9992	0·9992	0·9992	0·9992	0·9993	0·9993
3·2	0·9993	0·9993	0·9994	0·9994	0·9994	0·9994	0·9994	0·9995	0·9995	0·9995
3·3	0·9995	0·9995	0·9995	0·9996	0·9996	0·9996	0·9996	0·9996	0·9996	0·9997
3·4	0·9997	0·9997	0·9997	0·9997	0·9997	0·9997	0·9997	0·9997	0·9997	0·9998

★★★