322005

COMBINED COMPETITIVE EXAMINATION (MAIN)

MATHEMATICS

Paper–II

Time : 3 Hours *Full Marks* : 200

Note : *(1) The figures in the right-hand margin indicate full marks for the questions.*

(2) Question No. **1** *is compulsory.*

(3) Attempt ***four more questions from question nos.*** **2-10** *taking at least one question from each of the three sections— Section A, Section B, Section C.*

1. Answer any ***eight*** questions from the following : $5 \times 8 = 40$

(a) Prove that every subgroup of an Abelian group is normal.

(b) Show that every Cauchy sequence of real numbers is bounded.

(c) Test for convergence of the improper integral

$$\int_0^\infty e^{-mx} dx,\ m > 0.$$

(d) Show that $f(z) = Re(z)$ is nowhere differentiable.

(e) Form a partial differential equation for $\phi(x + y + z,\ x^2 + y^2 - z^2) = 0$ where ϕ is an arbitrary function.

(f) Determine the complex potential due to a source of strength K situated at $z = Z$.

(g) Convert the hexadecimal numbers FA, 2A3E, FFFF, 5A0E9, CB1E to decimal numbers.

(h) Find the mean deviation of the numbers 31, 35, 29, 63, 55, 72, 37 about the arithmetic mean.

(i) Obtain a basic solution of the system of equation :

$$x + 2y + z = 4$$
$$2x + y + 5z = 5$$

(j) Explain the Maximin-Minimax Principle for the selection of the optimal strategies by the two-players in a zero-sum game.

SECTION-A

2. Answer any ***four*** questions from the following : 10×4=40

(a) If G is a group having no non-trivial subgroups, then show that G must be finite having prime order.

(b) Show that every homomorphic image of a group G is isomorphic to a quotient group of G.

(c) Let R be the ring of 2×2 matrices over the integers and $A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z} \right\}$. Show that A is a right ideal of R but not a left ideal of R.

(d) Show that a Euclidean domain is a principal ideal domain but the converse is not true.

(e) Show that an integral domain R with unity is a UFD (unique factorization domain) if and only if every non-zero non-unit element is finite product of primes.

(f) Show that a finite extension of a field is an algebraic extension.

3. Answer any ***four*** questions from the following : 10×4=40

(a) Suppose A is a compact subset of the set of real numbers R and $f : A \to$ R is a continuous map. The show that f is uniformly continuous.

(b) Show that a bounded sequence of real numbers has a convergent subsequence. Give an example to show that an unbounded sequence may also have infinitely many convergent subsequences.

(c) If f is a bounded and Riemann integrable function defined on $[a, b]$, then show $|f|$ is also Riemann integrable on $[a, b]$ and

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx$$

(d) Show that the function $f :$ R $\to$ R defined by $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$ but the partial derivatives f_x and f_y both exist at origin and are equal.

(e) Prove that every absolutely convergent series of real numbers is convergent. Use it to show that the series

$$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$$

is convergent.

(f) Evaluate $\iint xy(x+y)\ dxdy$ over the region bounded by the curves $y = x^2$ and $y = x$.

4. Answer any ***four*** questions from the following : 10×4=40

(a) If a function $f(z)$ is analytic at a point z_0, then show that the Cauchy-Riemann equations are satisfied at each point of a neighbourhood containing z_0.

(b) Evaluate

$$\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$$

where C is the positively oriented circle $|z| = 1$.

(c) Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for $0 < |z-2| < 1$

(d) If C is a positively oriented simple closed contour and f is analytic inside and on C except for a finite number of singular points $z_1, z_2, \cdots z_n$ inside C, then show that

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z)$$

where $Res_{z=z_k} f(z)$ denotes the residue of f at z_k.

(e) Show that $z = -\frac{1}{2}$ is a pole of the function $f(z) = \left(\frac{z}{2z+1}\right)^3$. Determine the order of the pole and calculate the corresponding residue.

(f) If f is analytic function in a domain D containing z_0 and if $f'(z_0) \neq 0$, then show that $w = f(z)$ is a conformal mapping at z_0.

SECTION-B

5. Answer any ***two*** questions from the following : 20×2=40

(a) Find the integral surface to the PDE

$$(x-y)y^2 \frac{\partial z}{\partial x} + (y-x)x^2 \frac{\partial z}{\partial y} = (x^2+y^2)z$$

through the curve $xz = a^3$, $y = 0$.

(b) Find complete integrals of the following PDE by Charpit's method :

$$2\left(z+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=y\left(\frac{\partial z}{\partial x}\right)^2$$

(c) Solve the following PDEs :

(i) $3\frac{\partial^2 z}{\partial x\partial y}-2\frac{\partial^2 z}{\partial y^2}-\frac{\partial z}{\partial y}=\sin(2x+3y)$

(ii) $\frac{\partial^2 z}{\partial x^2}-\frac{\partial z}{\partial y}=2y-x^2$

(d) Solve the two-dimensional Laplace equation $\nabla^2 u=0,\ 0\le x\le a,\ 0\le y\le b$ satisfying the boundary conditions $u(0, y)=0$, $u(x, 0)=0$, $u(x, b)=0$ and $\frac{\partial u}{\partial x}(a, y)=T\sin^3\frac{\pi y}{a}$.

6. Answer any *two* questions from the following : 20×2=40

(a) Derive Lagrange's equations for a holonomic dynamical system in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}=Q_r,\ r=1, 2, \cdots, n$$

where T is the kinetic energy of the system at time t when the system is specified by the n generalized co-ordinates $q_1, q_2, \cdots q_n$ $Q_1, Q_2, \cdots Q_n$ are generalized forces.

(b) Prove that the moment of inertia of a body of mass M about a line passing through the origin and having direction cosines l, m, n is $Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm$, where A, B, C are the moments of intertia and D, E, F are products of inertia about the coordinate axes.

(c) Give the physical significance implied in the equation of continuity in fluid motion.

A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\phi(vt-x)$. Prove that the velocity u (at time t and distance x from the origin) is given by

$$v+\frac{(u_0-v)\,\phi(vt)}{\phi(vt-x)}$$

(d) Liquid is contained between two parallel plates; the free surface is a circular cylinder of radius a whose axis is perpendicular to the plates. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove that if Π be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is

$$\Pi \frac{\log r - \log b}{\log a - \log b}$$

7. Answer any ***four*** questions from the following : 10×4=40

(a) Determine $y(0.2)$ using the fourth order Runge-Kutta method for the IVP

$$y' = (x + y) \sin x, \; y(0) = 5.$$

(b) Using Newton-Raphson method, find a root, correct to four decimal places, of the equation $xe^x - 1 = 0$ which is close to 0.5.

(c) Write down the formula for composite Simpson's rule and the associated error bound.

Use composite Simpson's rule with four sub-intervals to approximate $\int_0^1 \frac{dx}{3+2x}$. Also find the error bound.

(d) What are the three schemes for representation of a signed integer? Describe each of these schemes and elaborate taking the integer – 25.

(e) Write an algorithm and draw the corresponding flow chart for finding a root of the real continuous function f defined on the interval $[a, b]$ with $f(a) f(b) < 0$.

(f) Write a program in BASIC to find the cube root of an integer using

(i) Sub-procedure, *(ii)* Function procedure.

SECTION-C

8. Answer any ***four*** questions from the following : 10×4=40

(a) For any two events A and B of a sample space S show that

$$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$$

(b) A bag contains 5 balls and it is not known how many of them are red. Two balls are drawn and are found to be red. What is the probability that all the balls are red?

(c) A random variable X has the density functions

$$f(x)=\begin{cases} cx^2, & 1\le x\le 2 \\ cx, & 2<x<3 \\ 0, & Otherwise \end{cases}$$

Find the value of the constant c. Also find $P(X > 2)$.

(d) The joint probability function of two discrete random variables X and Y is given by -

$f(x, y) = \frac{1}{36}\, xy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$ and zero otherwise. Find the marginal probability functions of X and Y. Also determine whether X and Y are independent.

(e) A random variable X has the density function

$$f(x)=\begin{cases} e^{-x} & x\ge 0 \\ 0, & otherwise \end{cases}$$

Find the value of $E[(X-1)^2]$.

(f) Prove that the mean and variance of a binomially distributed random variable X with parameters p and n, are np and $np(1-p)$ respectively.

9. Answer any *two* questions from the following : 20×2=40

(a) Find the second, third and fourth moments of the frequency distribution given below. Hence find a measure of skewness and kurtosis of the given distribution :

Class limits	Frequency
100-104.9	7
105-109.9	13
110-114.9	25
115-119.9	25
120-124.9	30
Total	100

(b) Find the sampling distribution of the mean of a sample of size n, drawn from a χ^2-population with n degrees of freedom.

(c) Solve the following 2-person zero-sum game :

$$\begin{array}{cc} & \textit{Player } B \\ \textit{Player } A & \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 0 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

10. Answer any ***two*** questions from the following : 20×2=40

(a) Use Simplex method to solve the following LPP :

Maximize $Z = 2x_1 - x_2 + x_3$, subject to the constraints

$$3x_1 + x_2 + x_3 \le 60,$$
$$x_1 - x_2 + 2x_3 \le 10,$$
$$x_1 + x_2 - x_3 \le 20,$$
$$x_1, x_2, x_3 \ge 0,$$

(b) Show that the dual of the dual of a linear programming problem (LPP) is the primal itself. Also verify this result for the LPP :

Minimize $Z = 15x + 10y$, subject to

$$3x + 5y \ge 5,$$
$$5x + 2y \ge 3, \; x, y \ge 0.$$

(c) Consider the problem of assigning five jobs to five persons. The assignment cost are given as follows:

Job

		1	2	3	4	5
Person	*A*	8	4	2	6	1
	B	0	9	5	5	4
	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Determine the optimal assignment schedule.

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