COMBINED COMPETITIVE EXAMINATION (MAIN)

MATHEMATICS

Paper—II

Time : 3 hours *Full Marks* : 200

Note : *(1) The figures in the right-hand margin indicate full marks for the questions.*

*(2) Attempt **five** questions in all.*

*(3) Question No. **1** is compulsory.*

1. Answer any *ten* questions : 4×10=40

(a) Prove that the function $f(x)=\frac{1}{x}$ is continuous on (0, 1) but not uniformly continuous.

(b) Using differentials, find an approximate value of the square root $\sqrt{25\cdot 2}$ (up to four decimal places).

(c) Prove that a group G is Abelian if every element of G (except the identity e) is of the order two.

(d) If H is a subgroup of G and N is a normal subgroup of G, show that $H \cap N$ is a normal subgroup of H.

(e) Form the partial differential equation by eliminating a, b from the curve $2z=(ax+y)^2+b$.

(f) Show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin although the Cauchy-Riemann equations are satisfied at that point.

(g) Find the bilinear transformation which maps the points $z_1=2$, $z_2=i$ and $z_3=-2$ into the points $w_1=1$, $w_2=i$ and $w_3=-1$ respectively.

(h) Convert the following binary numbers to decimal equivalents :

(i) 111100

(ii) 111111

(*i*) Convert the following hexadecimal numbers to their decimal equivalents :

(*i*) F117

(*ii*) EBA.C

(*j*) Find the mathematical variance of the sum of points on n dice.

(*k*) If A and B are any two events, then show that $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.

(*l*) If A and B are subsets of R which are non-empty and bounded below, then prove that $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

2. Answer any *eight* questions : 5×8=40

(*a*) In an LPP, if the objective function $f(x)$ attains its maximum at an interior point of P_F, then show that f is constant provided P_F is bounded.

(*b*) Prove that a hyperplane in R^n is a closed convex set.

(*c*) Write a flowchart to the first n-Fibonacci numbers.

(*d*) A card is drawn from a well-shuffled pack of 52 cards. Find the probability that the drawn card is neither a spade nor a jack.

(*e*) If $f = (1\ 2\ 3\ 4\ 5)$, then find f^{21}.

(*f*) Prove that any two right (left) cosets of a sub-group are either disjoint or identical. Also show that $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

(*g*) Prove that an absolutely convergent series is convergent.

(*h*) Prove that the series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \quad (x \geq 0)$ is convergent for $0 \leq x < 1$ and divergent for $x \geq 1$.

(*i*) Find the general solution of the differential equation $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$.

(*j*) A function $f(x)$ is defined as follows :

$$f(x) = \frac{x}{1+e^{\frac{1}{x}}}, \quad \text{when } x \neq 0$$
$$f(0) = 0, \quad \text{when } x = 0$$

Show that $f(x)$ is not differentiable at $x = 0$.

3. Answer any *five* questions : 8×5=40

(a) Find a complete integral of the equation $p^2x+q^2y=z$ by Charpit's method.

(b) Show that a bounded function f which has only a finite number of points of discontinuity in $[a, b]$ is integrable in $[a, b]$.

(c) Prove that a ring R has no divisors of zero if and only if the cancellation laws hold in R.

(d) Find all the maxima and minima of the function given by $f(x, y)=x^3+y^3-63(x+y)+12xy$.

(e) Prove that a group of order p^2 is Abelian.

(f) Find the number of real roots of the equation $2x^5-4x^4-9x-2=0$ and determine the pair of consecutive integers between which they lie.

(g) Show that the function $u=x^3-3xy^2$ is harmonic and find the corresponding analytic function.

4. Answer any *four* questions : 10×4=40

(a) Prove that the order of each sub-group of a finite group G is a divisor of the order of the group G. Also show that its converse is not true.

(b) Calculate the approximate value of $\int_0^{\frac{\pi}{2}} \sin x\, dx$ by

(i) trapezoidal rule;

(ii) Simpson's rule using 11 ordinates.

Also find which method gives greater accuracy.

(c) Find by Newton-Raphson method, correct to six places of decimals, the root of the equation

$$x \log_{10} x = 4 \cdot 772393$$

(d) Prove that every field is an integral domain.

(e) Find the maximum and minimum values, if they exist, of the function $f(x, y)=x-3y$ where x and y are non-negative and are subject to the inequalities $3x+4y \geq 19$, $2x-y \leq 9$, $2x+y \leq 15$ and $x-y \geq -3$.

5. Answer any *two* questions : 20×2=40

(a) State and prove Cauchy's integral theorem.

(b) Obtain the general solution of the equation $y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = xy$.

(c) If $f(x, y) = (x^2 + y^2)\tan^{-1}\frac{y}{x}$ when $x \neq 0$ and $f(0, y) = \frac{\pi y^2}{2}$, show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

6. Answer any *four* questions : 10×4=40

(a) Evaluate $\int_L \frac{dz}{z}$, where L represents the square described in the positive sense with sides parallel to the axes and of length $2a$ and having its centre at the origin.

(b) Show that if G be a finite Abelian group and a prime p divides order of G, then G has an element of order p.

(c) Show that the moment of inertia of a lamina in the shape of an isosceles right-angled triangle about its hypotenuse is $\frac{Ma^2}{24}$, where M is its mass and a, the length of the hypotenuse.

(d) If X is a uniformly distributed random variable over the interval (1, 4), find the probability that $Y < 0$, where $Y = x^2 - 4$.

(e) Explain the principles of dominance in game theory.

7. Answer any *two* questions : 20×2=40

(a) Using D'Alembert's principle, derive Hamilton's principle.

(b) Define holonomic and non-holonomic systems. Also deduce the principle of energy from the Lagrange's equation of motion.

(c) The particles of a fluid move symmetrically in space with regard to a fixed centre, prove that the equation of continuity is $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0$, where u is the velocity at a distance r.

8. (a) State Newton-Gregory forward interpolation and backward interpolation formulas. If $\log_{10} 654 = 2 \cdot 8156$, $\log_{10} 658 = 2 \cdot 8182$, $\log_{10} 659 = 2 \cdot 8189$ and $\log_{10} 661 = 2 \cdot 8202$, find $\log_{10} 656$.

(b) Let X and Y be two Bernoulli random variables with the same parameter $p = \frac{1}{2}$. Can the support of their sum be equal to $\{0, 1\}$? What about the case where p is not necessarily equal to $\frac{1}{2}$? Note that no particular dependence structure between X and Y is assumed.

20+20=40

9. (a) Find the dual of the following linear programming problem :

Minimize $Z = 2x_1 + 3x_2 + 4x_3$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &\geq 2 \\ 3x_1 + x_2 + 7x_3 &= 3 \\ x_1 + 4x_2 + 6x_3 &\leq 5 \\ x_1, \ x_2 &\geq 0 \end{aligned}$$

and x_3 is unrestricted. Also prove that the dual of the dual is primal.

(b) Define a balanced transportation. Prove that it has at least one feasible solution.

20+20=40

10. (a) Show that the transformation $w = i \frac{1-z}{1+z}$ transforms the circle $|z| = 1$ onto the real axis of the w-plane and the interior of the circle into the upper half of the w-plane.

(b) Apply Gauss-Seidel iteration method to solve the following equations :

$$\begin{aligned} 10x_1 + x_2 + x_3 &= 12 \\ 2x_1 + 10x_2 + x_3 &= 13 \\ 2x_1 + 2x_2 + 10x_3 &= 14 \end{aligned}$$

20+20=40

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