COMBINED COMPETITIVE EXAMINATION (MAIN)

MATHEMATICS–I

Time : 3 hours

Full Marks : 200

Note : *(1) The figures in the right-hand margin indicate full marks for the questions.*

*(2) Attempt **five** questions in all.*

*(3) Question No. **1** is compulsory.*

1. Answer any *ten* of the following :

4×10=40

(a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

(b) Prove that

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

(c) Show that

$$B(m, n) = B(m+1, n) + B(m, n+1)$$

(d) What are direction cosines of lines equally inclined to the axes? How many such lines are there?

(e) Find the singular solution of

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

(f) Prove that the necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant magnitude is

$$\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$$

(g) Prove that if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their heights.

(h) A rod of length l is lying in the xy-plane. Write down the equations of constraints, and calculate the degrees of freedom.

(i) Prove that the pressure of a heavy homogeneous fluid at all points in the same horizontal plane is the same.

(j) Suppose that the half-life of a certain particle is 10^{-7} second; when it is at rest. What will be its half-life when it is travelling with a speed of $0 \cdot 99c$?

(k) In an orbit described under a central force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.

(l) Prove that Kronecker delta is an invariant tensor.

2. Answer any *eight* of the following : 5×8=40

(a) If

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

then prove that

$$A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$$

where k is any positive integer.

(b) Find whether the vectors

$$2x^3 + x^2 + x + 1, \ x^3 + 3x^2 + x - 2, \ x^3 + 2x^2 - x + 3$$

of $R[x]$, the vector space of all polynomials over the real number field R, are linearly dependent or not.

(c) Test the continuity of the following function :

$$f(x) = \begin{cases} \frac{1}{(x-a)}\operatorname{cosec}\left(\frac{1}{x-a}\right) & , \text{ when } x \neq a \\ 0 & , \text{ when } x = a \end{cases}$$

(d) Solve :

$$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$$

(e) Show that contraction of a mixed tensor of order two is an invariant.

(f) Find the equation of a plane through the points (2, 2, 1) and (9,3,6), and perpendicular to the plane $2x + 6y + 6z = 9$.

(g) If

$$u = \sin^{-1}\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

then show that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}$$

(h) Show that the improper integral

$$\int_0^1 x^{n-1}e^{-x}dx$$

is convergent, if $n > 0$.

(i) A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be

$$\frac{v^2 c}{s^2 + c^2}$$

then prove that the curve is a catenary.

(j) Find the least force required to pull a body on a rough horizontal plane.

3. Answer any *five* of the following : 8×5=40

(a) Reduce the equation

$2x^2 + 7y^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 2 = 0$

to a canonical form.

(b) Reduce the matrix

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 8 \\ 3 & 6 & 12 \end{bmatrix}$$

to echelon form, and hence find the rank of the matrix A.

(c) Show that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-4}{5} = \frac{y-1}{2} = z$$

intersect. Find their point of intersection.

(d) Find the radius of the circular section of the sphere $[\vec{r}] = 5$ by the plane

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$$

(e) Prove that every differentiable function is continuous, but every continuous function need not be differentiable.

(f) Use Taylor's theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1} x + h\sin z \frac{\sin z}{1} - (h\sin z)^2 \frac{\sin 2z}{2!} + (h\sin z)^3 \frac{\sin 3z}{3!} - \dots$$

where $z = \cot^{-1} x$.

(g) A horizontal shelf is moved up and down with SHM of period $\frac{1}{2}$ second. What is the greatest amplitude admissible in order that a weight placed on the shelf may not be jerked off ?

4. Answer any *four* of the following : 10×4=10

(a) If A_i is a vector, then show that, in general, $\frac{\partial A_i}{\partial x^k}$ is not a tensor, but that $\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$ is a tensor.

(b) Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines, if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

(c) Show that the semivertical angle of the right cone of given total surface and maximum volume is $\sin^{-1}\frac{1}{3}$.

(d) Solve

$$y_3 - 6y_2 + 11y_1 - 6y = e^{2x}$$

by the method of variation of parameters.

(e) Solve :

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

5. Answer any *two* of the following : 20×2=40

(a) (i) Show that

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$$

Find the values of the two integrals.

(ii) If

$$I_{m,n} = \int \cos^m x \sin^n x \, dx$$

then show that

$$(m+n)I_{m,n} = \cos^{m-1} x \sin^{n+1} x + (m-1)I_{m-2,\, n}$$

(b) (i) Prove that

$$\text{div grad } r^n = \nabla^2 r^n = n(n+1)r^{n-2}$$

(ii) If S be a closed surface and $\vec{r}$ be the position vector of any point (x, y, z) measured from the origin O, then show that

$$\int_S \frac{\vec{r}}{r^3} \cdot \hat{n}\, dS = \begin{cases} 0, & \text{if } O \text{ lies outside } S \\ 4\pi, & \text{if } O \text{ lies inside } S \end{cases}$$

(c) Find the characteristic roots and the corresponding characteristic vectors for the following matrix :

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

6. Answer any *four* of the following : 10×4=40

(a) Prove that $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$ represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$. Find also the equations of the axis of the cylinder.

(b) Prove that

$$m = m_0 \Big/ \left(1 - \frac{u^2}{c^2}\right)^{1/2}$$

where u is velocity of the body when its mass is m and m_0 is the mass of the body when it is at rest.

(c) Show that the set

$$\{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$$

forms a basis for $V_3(F)$.

(d) Prove that if U is a subspace of a finite-dimensional vector space $V(F)$, then

$$\dim(V/U) = \dim V - \dim U$$

V/U being a quotient space of V modulo U.

(e) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. In what direction the directional derivative will be maximum and what is its magnitude? Also find the unit normal vector and tangent plane to the surface $x^2yz + 4xz^2 = 6$ at the point $(1, -2, -1)$.

7. Answer any *two* of the following : 20×2=40

(a) A uniform rod rests in a vertical plane within a fixed hemispherical bowl whose radius is equal to the length of the rod. If μ is the coefficient of friction between the rod and the bowl, then show that in limiting equilibrium the inclination of the rod to the horizontal is $\tan^{-1}[4\mu/(3-\mu^2)]$.

(b) A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest; prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

(c) A particle is projected vertically upwards from the earth's surface with velocity just sufficient to carry it to infinity. Prove that the time it takes in reaching a height h is

$$\frac{1}{3}\sqrt{\frac{2a}{g}}\{(1+\frac{h}{a})^{3/2}-1\}$$

where a is the earth's radius.

8. Answer the following questions : 20+20=40

(a) A uniform beam of length l rest with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β ($\beta > \alpha$), then show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$$

and show that the beam is unstable in this position.

(b) A uniform chain of length l and weight W hangs between two fixed points at the same level and weight W' is attached at the middle point. If k be the sag in the middle, then prove that the pull on either point of support is

$$\frac{k}{2l}W + \frac{l}{4k}W' + \frac{l}{8k}W$$

9. Answer the following question : 10+20+10=40

(a) Use Lagrange's mean value theorem to show that

$$\frac{x}{1+x} < \log(1+x) < x, \text{ for every } x > 0$$

Deduce that $0 < \dfrac{1}{\log(1+x)} - \dfrac{1}{x} < 1.$

(b) If $x = r\cos\theta \cdot \cos\phi,\ y = r\sin\theta\sqrt{(1-m^2\sin^2\phi)},\ z = r\sin\phi(1-n^2\sin^2\theta)$, where $m^2 + n^2 = 1$, then prove that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \frac{r^2(m^2\cos^2\phi + n^2\cos^2\theta)}{[(1-m^2\sin^2\phi)(1-n^2\sin^2\theta)]^{1/2}}$$

(c) Find the position of the centre of gravity of an arc of a circle of radius a, which subtends an angle 2α at the centre.

10. Answer the following questions : 20+20=40

(a) *(i)* If A^{ij} is a skew-symmetric tensor, then show that

$$\frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}A^{ij})}{\partial x^i}$$

is a tensor.

(ii) Evaluate the Christoffel's symbols for the spaces, where $g_{ij} = 0$ if $i \neq j$.

(b) A triangular lamina ABC of density ρ floats in a liquid of density σ with its plane vertical, the angle B being in the surface of the liquid, and the angle A not immersed. Show that

$$\frac{\rho}{\sigma} = \frac{\sin A \cos C}{\sin B} = \frac{a^2 + b^2 - c^2}{2b^2}$$

a, b, c being the lengths of the sides of the triangle.

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